# Advanced Calculus — Final Exam

June 25, 2007

6% 1. (a) Show that the improper integral $\int_1^\infty \frac{\cos x}{\sqrt{x}}dx$ is convergent.

**Solution:** Since $|\int_1^c \cos x dx| \le 2$ for any $c \ge 1$, and $\frac{1}{\sqrt{x}}$ is decreasing on $[1,\infty)$, $\int_1^\infty \frac{\cos x}{\sqrt{x}}dx$ converges by Dirichlet test.

6% (b) Show that $\int_0^\infty e^{-x^2}\cos tx\, dx$ converges uniformly for all $t \in \mathbb{R}$.

**Solution:** Since $|e^{-x^2}\cos tx| \le e^{-x^2}$, for $x \in [0,\infty)$, and $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} < \infty$, $\int_0^\infty e^{-x^2}\cos tx\, dx$ converges uniformly by Weierstrass M-test.

8% 2. Let $G$ be defined for $t > 0$ by $G(t) = \int_0^\infty e^{-x^2}\cos tx\, dx$. Show that $G(t) = \frac{\sqrt{\pi}}{2}e^{\frac{-t^2}{4}}$.

**Solution:** Note that $G(t) = 2\int_0^\infty e^{-x^2}\sin tx\, dx = \frac{1}{2}e^{-x^2}\sin tx|_0^\infty - \frac{1}{2}\int_0^\infty te^{-x^2}\cos tx\, dx = -\frac{t}{2}G(t)$. Hence, $G(t) = Ce^{\frac{-t^2}{4}}$, and $C = G(0) = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

3. Let $f \in PC(2\pi)$ and $T_n(x) = \frac{1}{2}\alpha_0 + \sum_{k=1}^n \left(\alpha_k \cos kx + \beta_k \sin kx\right)$.

10% (a) Show that

$$\|f - T_n\|_2^2 = \|f\|_2^2 - \pi\left\{\frac{1}{2}a_0^2 + \sum_{k=1}^n \left(a_k^2 + b_k^2\right)\right\} + \pi\left\{\frac{1}{2}(\alpha_0 - a_0)^2 + \sum_{k=1}^n \left[(\alpha_k - a_k)^2 + (\beta_k - b_k)^2\right]\right\}$$

where $a_k$, $b_k$ denote the Fourier coefficients of $f$.

**Solution:** By using a direct calculation, we can obtain the above inequality.

4% (b) Show that $\frac{1}{2}a_0^2 + \sum_{k=1}^n \left(a_k^2 + b_k^2\right) \le \frac{1}{\pi}\|f\|_2^2$.

**Solution:** Let $T_n = S_n(f)$ in the inequality obtained in $(a)$, i.e. the $n$th degree Fourier polynomial of $f$. Since $\|f - S_n(f)\|_2^2 \ge 0$, we get the above Bessel's inequality.

8% (c) Let $g \in PC(2\pi)$. Show that $\lim_{n\to\infty}\int_{-\pi}^{\pi} g(t)\sin(n+\frac{1}{2})t\, dt = 0$.

**Solution:** Note that $\int_{-\pi}^{\pi} g(t)\sin(n+\frac{1}{2})t\, dt = \int_{-\pi}^{\pi}\left[g(t)\cos(\frac{t}{2})\right]\sin nt dt + \int_{-\pi}^{\pi}\left[g(t)\sin(\frac{t}{2})\right]\cos nt dt$, and the last two terms denote the Fourier coefficients of $g(t)\cos(\frac{t}{2})$, and $g(t)\sin(\frac{t}{2})$, respectively. We can extend $g(t)\cos(\frac{t}{2})$, $g(t)\sin(\frac{t}{2})$ periodically so that $g(t)\cos(\frac{t}{2}), g(t)\sin(\frac{t}{2}) \in PC(2\pi)$. Bessel's inequality in $(b)$ implies that $\frac{1}{2}a_0^2 + \sum_{k=1}^\infty \left(a_k^2 + b_k^2\right) < \infty$ holds for both $g(t)\cos(\frac{t}{2})$, and $g(t)\sin(\frac{t}{2})$. Hence, we have $\lim_{n\to\infty} a_n = 0$, and $\lim_{n\to\infty} b_n = 0$ for both functions, i.e. we have $\lim_{n\to\infty}\int_{-\pi}^{\pi}\left[g(t)\cos(\frac{t}{2})\right]\sin nt dt = 0$, and $\lim_{n\to\infty}\int_{-\pi}^{\pi}\left[g(t)\sin(\frac{t}{2})\right]\cos nt dt = 0$.

10% 4. (a) Let $f \in PC(2\pi)$ be such that $f(x) = x$ for $x \in (-\pi,\pi]$. Find the Fourier series of $f$.

**Solution:** Since $f$ is odd, $a_n = 0$ for all $n$, and its Fourier series is $2\sum_{n=1}^\infty \frac{(-1)^{n-1}\sin nx}{n}$.

6% (b) Use Parseval's Equality to establish that $\frac{\pi^2}{6} = \sum_{n=1}^\infty \frac{1}{n^2}$.

**Solution:** $\dfrac{2\pi^2}{3} = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x^2 = \dfrac{1}{\pi}\|f\|_2^2 = \dfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2) = 4\sum_{n=1}^{\infty}\dfrac{1}{n^2}$.

$\boxed{6\%}$ 5. Use Inverse Function Theorem to determine whether the system

$$\begin{aligned} u(x,y,z) &= x+xyz \\ v(x,y,z) &= y+xy \\ w(x,y,z) &= z+2x+3z^2 \end{aligned}$$

can be solved for $x,y,z$ in terms of $u,v,w$ near $p=(0,0,0)$.

**Solution:** Set $F(x,y,z)=(u,v,w)$. Then $DF(p) = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}(p) = \begin{bmatrix} 1+yz & xz & xy \\ y & 1+x & 0 \\ 2 & 0 & 1+6z \end{bmatrix}(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 1 \neq 0$. By the Inverse Function Theorem, the inverse $F^{-1}(u,v,w)$ exists near $p=(0,0,0)$, i.e. we can solve $x,y,z$ in terms of $u,v,w$ near $p=(0,0,0)$.

6. Let $F:\mathbb{R}^4 \to \mathbb{R}^2$ be given by $F(x,y,z,w) = (G(x,y,z,w), H(x,y,z,w)) = (y^2+w^2-2xz, y^3+w^3+x^3-z^3)$, and let $p=(1,-1,1,1)$.

$\boxed{6\%}$ (a) Show that we can solve $F(x,y,z,w)=(0,0)$ for $(x,z)$ in terms of $(y,w)$ near $(-1,1)$.

**Solution:** Since $DF(p) = \begin{bmatrix} G_x & G_y & G_z & G_w \\ H_x & H_y & H_z & H_w \end{bmatrix}(p) = \begin{bmatrix} -2 & -2 & -2 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$ and $\begin{vmatrix} G_x & G_z \\ H_x & H_z \end{vmatrix}(p) = \begin{vmatrix} -2 & -2 \\ 3 & -3 \end{vmatrix} = 12 \neq 0$, we can write $(x,z)$ in terms of $(y,w)$ near $(-1,1)$ by Implicit Function Theorem.

$\boxed{8\%}$ (b) If $(x,z) = \Phi(y,w)$ is the solution in part $(a)$, show that $D\Phi(-1,1)$ is given by the matrix

$$-\begin{bmatrix} -2 & -2 \\ 3 & -3 \end{bmatrix}^{-1}\begin{bmatrix} -2 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Solution:** The Implicit Function Theorem implies that $F(x,y,z,w)=(0,0)$ near $p$ if and only if $(x,z)=\Phi(y,w)$ near $(-1,1)$. Hence, we have $\dfrac{\partial F}{\partial y} = (0,0)$, and $\dfrac{\partial F}{\partial w} = (0,0)$ near $(-1,1)$.

Therefore, $0 = G_x\dfrac{\partial x}{\partial y} + G_y + G_z\dfrac{\partial z}{\partial y}$, and $0 = G_x\dfrac{\partial x}{\partial w} + G_z\dfrac{\partial z}{\partial w} + G_w$,

which implies that $-[G_y, G_w] = [G_x, G_z]\begin{bmatrix} \dfrac{\partial x}{\partial y} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial z}{\partial y} & \dfrac{\partial z}{\partial w} \end{bmatrix}$.

Similarly, we have $-[H_y, H_w] = [H_x, H_z]\begin{bmatrix} \dfrac{\partial x}{\partial y} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial z}{\partial y} & \dfrac{\partial z}{\partial w} \end{bmatrix}$.

Thus, we have $-\begin{bmatrix} G_y & G_w \\ H_y & H_w \end{bmatrix} = \begin{bmatrix} G_x & G_z \\ H_x & H_z \end{bmatrix}\begin{bmatrix} \dfrac{\partial x}{\partial y} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial z}{\partial y} & \dfrac{\partial z}{\partial w} \end{bmatrix}$

or $D\Phi = \begin{bmatrix} \dfrac{\partial x}{\partial y} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial z}{\partial y} & \dfrac{\partial z}{\partial w} \end{bmatrix} = -\begin{bmatrix} G_x & G_z \\ H_x & H_z \end{bmatrix}^{-1}\begin{bmatrix} G_y & G_w \\ H_y & H_w \end{bmatrix}$

Hence, $D\Phi(-1,1)$ is given by the matrix

$$-\begin{bmatrix} -2 & -2 \\ 3 & -3 \end{bmatrix}^{-1}\begin{bmatrix} -2 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. Let $f_n : [0,1] \to \mathbb{R}$ be continuous and be such that $|f_n(x)| \le 100$ for every $n$ and for all $x \in [0,1]$ and the derivatives $f_n'(x)$ exist and are uniformly bounded on $(0,1)$.

$\boxed{6\%}$ (a) Show that there is a constant $M$ such that $|f_n(x) - f_n(y)| \le M|x-y|$ for any $x, y \in [0,1]$ and any $n \in \mathbb{N}$.

**Solution:** Let $M$ be a constant such that $|f_n'(x)| \le M$ for all $x \in (0,1)$. By the mean value theorem, we get $|f_n(x) - f_n(y)| \le M|x-y|$ for any $x, y \in [0,1]$ and any $n \in \mathbb{N}$.

$\boxed{8\%}$ (b) Prove that $f_n$ has a uniformly convergent subsequence.
[**Hint :** You may want to use Arzela-Ascoli Theorem to prove this.]

**Solution:** We apply the Arzela-Ascoli Theorem by verifying that $\{f_n\}$ is equicontinuous and bounded. Given $\varepsilon$, we can choose $\delta = \varepsilon/M$, independent of $x$, $y$, and $n$. Thus $\{f_n\}$ is equicontinuous. It is bounded because $\|f_n\| = \sup\limits_{x\in[0,1]} |f_n(x)| \le 100$.

$\boxed{8\%}$ 8. Let the functions $f_n : [a,b] \to \mathbb{R}$ be uniformly bounded continuous functions. Set $F_n(x) = \int_a^x f_n(t)\,dt$, for $x \in [a,b]$.
Prove that $F_n$ has a uniformly convergent subsequence.
[**Hint :** You may use #7 to prove this.]

**Solution:** Since $\|F_n\| \le \|f_n\|(b-a)$, $F_n$ is uniformly bounded. Also, since $|F_n'(x)| \le \|f_n\|$, $F_n$ is equicontinuous by the preceding result. Therefore, $F_n$ has a uniformly convergent subsequence by Arzela-Ascoli Theorem.